Chapter VI

A NEW ORDER IN SPACE ~ ASPECTS OF A THREEFOLD ORDERING OF THE FUNDAMENTAL SYMMETRIES OF EMPIRICAL SPACE, AS EVIDENCED IN THE PLATONIC AND ARCHIMEDEAN POLYHEDRA ~ TOGETHER WITH A TWOFOLD EXTENSION OF THE ORDER TO INCLUDE THE REGULAR AND SEMI-REGULAR TILINGS OF THE PLANAR SURFACE

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~ DEDICATED TO KEITH CRITCHLOW FRCA ~

ABSTRACT

There are generally considered to be five Platonic (Regular) and thirteen Archimedean (Semi-Regular) polyhedra (leaving aside the Kepler-Poinsot polyhedra, and disregarding the regular prisms and antiprisms).[1] These are important as they represent the most regular articulation that the space of our experience is capable of. But as far as I am aware, a fully satisfactory pattern which situates these polyhedra in their proper interrelationship has not previously been advanced.

Keith Critchlow proposes the tetrahedron and truncated tetrahedron as nuclear or 'over' solids, together with a twofold periodic arrangement of secondary regular 'parent' solids and their truncations for the remainder.[2] Cundy and Rollett propose a similar arrangement,[3] with the first two solids of tetrahedral symmetry, and the rest of octahedral or icosahedral symmetry. Coxeter suggests the Platonic solids fall naturally into two classes, the "crystallographic" solids (tetrahedron, octahedron and cube), and the "pentagonal" polyhedra (icosahedron, dodecahedron and Kepler-Poinsot solids).[4] But in my opinion these arrangements, although valuable, do not adequately integrate all the polyhedra into a rigorous and comprehensive order.

This writer proposes a threefold arrangement, which does appear rigorous and comprehensive. Thus the regular and semi-regular polyhedra are conveniently classified into three parallel sets, according to which one of only three spatial patterns of symmetries each exhibits: {2,3,3}-fold TetraTetrahedral, {2,3,4}-fold OctaHexahedral, or {2,3,5}-fold IcosiDodecahedral symmetry. These are the symmetries of the quasi-regular octahedron, cuboctahedron, and icosidodecahedron respectively. Four polyhedra exhibit two of these fundamental patterns, and accordingly reappear as different elements in either of the sets to which they belong. Two polyhedra each appear twice as different elements in the same set, in accord with their alternative orientations. Polyhedra which reappear are given alternative names. There are therefore in total three parallel sets of eight polyhedra each, making twenty-four polyhedra in all; there being then in effect six regular and eighteen semi-regular polyhedra.

Each element of a set strictly correlates with its corresponding elements in the other sets. Further, particular patterns - which are termed aspects - are replicated across each of the sets. Within each set of eight, subsets of elements are discernible according to various rules: sequential truncation, rotation and displacement, and octaving of faces. The particular aspect which characterises a subset of one set correlates with the respective aspects of the other sets. Properties of elements and of aspects can then be generalised across classes, and this predictive ability is confirmed in practice. A Gestalt pattern is advanced for each set which integrates the constituent aspects, and this is further developed to include the whole three sets.

The classification is also regularly extended to recognise a fourth and fifth class, which together comprise the three regular and all but one of the eight semi-regular tilings of the planar surface, (the exception $3^3.4^2$ being considered a degenerate case). These classes are of {2,3,6}-fold and {2,4,4}-fold symmetry, of the quasi-regular TriangularHexagonal Array and SquareSquare Array respectively. Within either two-dimensional class some elements are repeated with different colourings as different facial arrangements are associated with the respective symmetry pattern of that class, the repeated elements being given alternative names. Discounting the degenerate exception, there are then in effect a total of four regular and twelve semi-regular tilings of the plane. But they do not exhaust the thirty-two possible uniform colourings of regular and semi-regular tilings.

The harmony of interrelationship which characterises the entire order reveals the elegant structure of space. This research is from the author's forthcoming work on the Platonic and Archimedean Polyhedra, which will be available from the author.

THIS PAPER DEVELOPS KEITH CRITCHLOW'S WORK, *Order in Space,* and is hence respectfully described as a *"New Order in Space"*. He gives a progression of the regular polyhedra as a Regular Tetraktys, which pattern I do not address in this paper.

Critchlow then gives the following classification,[5] arranging the Archimedean Solids from left to right as a series of increasing truncations of the parent Octahedron or Icosahedron, that is as two classes of symmetry:

THE NUCLEAR or 'OVER' SOLIDS	THE SECONDARY REGULAR 'PARENT' SOLIDS	
Tetrahedron	Octahedron	Cube or Hexahedron
Truncated Tetrahedron	Icosahedron	Dodecahedron

THE ARCHIMEDEAN or SEMI-REGULAR SOLIDS: SIX TRUNCATIONS OF THE OCTAHEDRON AND OF THE ICOSAHEDRON					
Truncated Octahedron	*(Dymaxion)* Cuboctahedron	Truncated Cuboctahedron	Snub Hexahedron	Rhombic Cuboctahedron	Truncated Hexahedron
Truncated Icosahedron	IcosiDodecahedron	Truncated IcosiDodecahedron	Snub IcosiDodecahedron	Rhombic IcosiDodecahedron	Truncated Dodecahedron

I amend this by distinguishing three classes, which otherwise following Critchlow's schema would comprise successive truncations of the parent Tetrahedron, Octahedron, or Icosahedron:

THE SIX REGULAR PLATONIC 'PARENT' SOLIDS AS THREE POLAR PAIRS		
I	♂ Tetrahedron	♀ Tetrahedron
II	Octahedron	Hexahedron
III	Icosahedron	Dodecahedron

THE EIGHTEEN ARCHIMEDEAN or SEMI-REGULAR SOLIDS: SIX TRUNCATIONS OF THE TETRAHEDRON, OF THE OCTAHEDRON, AND OF THE ICOSAHEDRON					
Truncated ♂ Tetrahedron	*(Octahedron)* TetraTetrahedron	Great Rhombic TetraTetrahedron	Snub TetraTetrahedron	Small Rhombic TetraTetrahedron	Truncated ♀ Tetrahedron
Truncated Octahedron	*(Cuboctahedron)* OctaHexahedron	Great Rhombic OctaHexahedron	Snub OctaHexahedron	Small Rhombic OctaHexahedron	Truncated Hexahedron
Truncated Icosahedron	IcosiDodecahedron	Great Rhombic IcosiDodecahedron	Snub IcosiDodecahedron	Small Rhombic IcosiDodecahedron	Truncated Dodecahedron

But I develop this further, as the simple linear sequence is unsatisfactory. (For example, if the polyhedra are arranged in reverse as successive truncations of the Tetrahedron, Hexahedron, and Dodecahedron, the relative order of the rhombic and snub polyhedra changes). Coxeter properly places importance on the octahedron, cuboctahedron, and icosidodecahedron as Quasi-Regular solids.[6] I term these solids the TetraTetrahedron, OctaHexahedron, and IcosiDodecahedron, and recognise their significance as central elements of their respective classes. I have sought to use a consistent terminology, which necessitates long and rather clumsy terms. Further subsets are identified. The rich patterns are made apparent by contemplation of the accompanying illustrations.

THREE FUNDAMENTAL SYSTEMS OF POLYHEDRAL SYMMETRY AND TWO FUNDAMENTAL SYSTEMS OF POLYGONAL TILING SYMMETRY

The three classes of polyhedra and two classes of polygonal tiling are given by the integral three-dimensional patterns of the three Quasi-Regular polyhedra, and the integral two-dimensional patterns of what I postulate to be the two Quasi-Regular tilings. These are shown in Table 1 opposite.

The important quasi-regular elements from which these symmetry patterns are derived, shown in Fig 1, are termed the central or neutral elements of first degree. Kepler gave what is thought to be the first systematic treatment of the regular and semi-regular tilings of the plane, and considered them as analogues of the regular and semi-regular polyhedra.[7]

Class I	Class II	Class III	Class IV	Class V
the {2,3,3}-fold symmetry of the TetraTetrahedron	the {2,3,4}-fold symmetry of the OctaHexahedron	the {2,3,5}-fold symmetry of the IcosiDodecahedron	the {2,3,6}-fold symmetry of the Triangular Hexagonal Array	the {2,4,4}-fold symmetry of the SquareSquare Array (Chess Board)
(3-D)	*(3-D)*	*(3-D)*	*(2-D)*	*(2-D)*

Table 1: Fundamental Symmetry Patterns

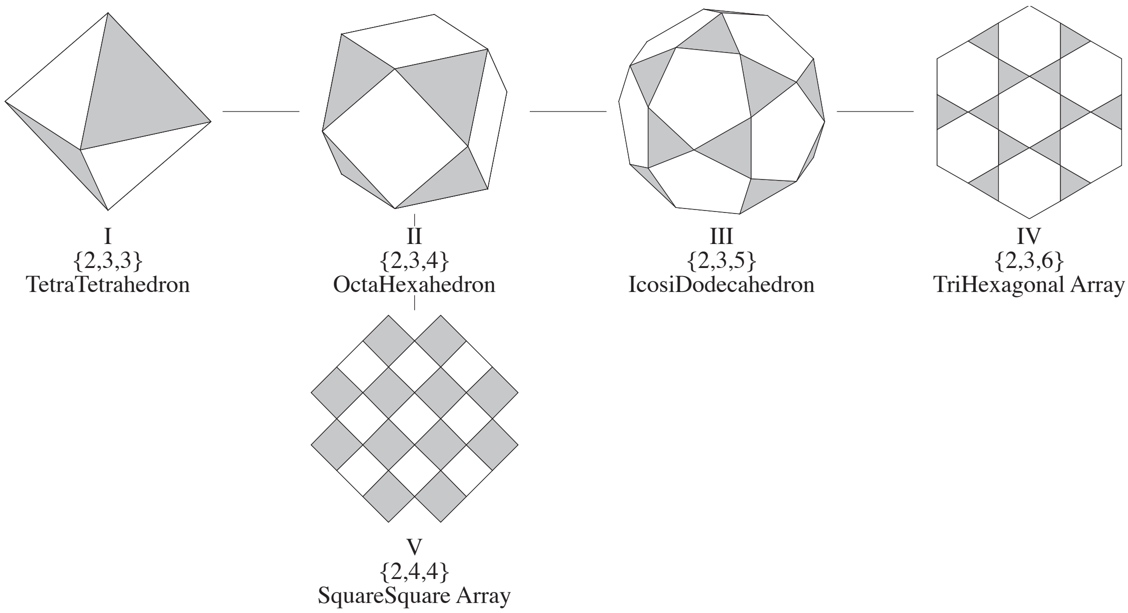


Figure 1 : Quasi-Regular elements from which the fundamental symmetry classes are derived.

In a similar spirit, I have extended the polyhedral order to include the two latter classes, which together generate the three regular and seven of the eight semi-regular tilings of the planar surface.[8] I omit an eighth semi-regular tiling $3^3.4^2$ as in my opinion it is degenerate.

The orderly progression of elements through the three polyhedral classes thus continues into a fourth polygonal class, which develops {2,3,6}-fold symmetry on the plane. The second polyhedral class is developed orthogonally to that fourfold sequence to yield the fifth polygonal class, which is of {2,4,4}-fold symmetry on the plane. There are appropriately three classes for three-dimensional space, and two classes for two-dimensional space.

In both classes IV and V, all axes are parallel and are perpendicular to the plane, rather than coincident at the centre of the polyhedra. It will therefore be appreciated that Classes I, II and III refer to the symmetries of regular three-dimensional polyaxial *centralised* solids of finite extent; whilst Classes IV and V refer to the symmetries of regular two-dimensional polyaxial *decentralised* tilings of infinite extent, (but which could be considered as centralised forms where the Centre is at infinity). Elsewhere I discuss three-dimensional polyaxial centralised zonahedral clusters,[9] which contrast with the decentralised space filling patterns described by Critchlow,[10] and in a separate work I fully document one such centralised expansion.[11]

Within the two-dimensional classes, elements reappear as for the polyhedral classes, ascribing different facial arrangements to the patterns of positive, neutral and negative symmetry axes. These correspond to different uniform colourings of the same semi-regular tiling. In general, aspects can be generalised from the first three classes to the other, with common-sense being exercised in the change in dimension.

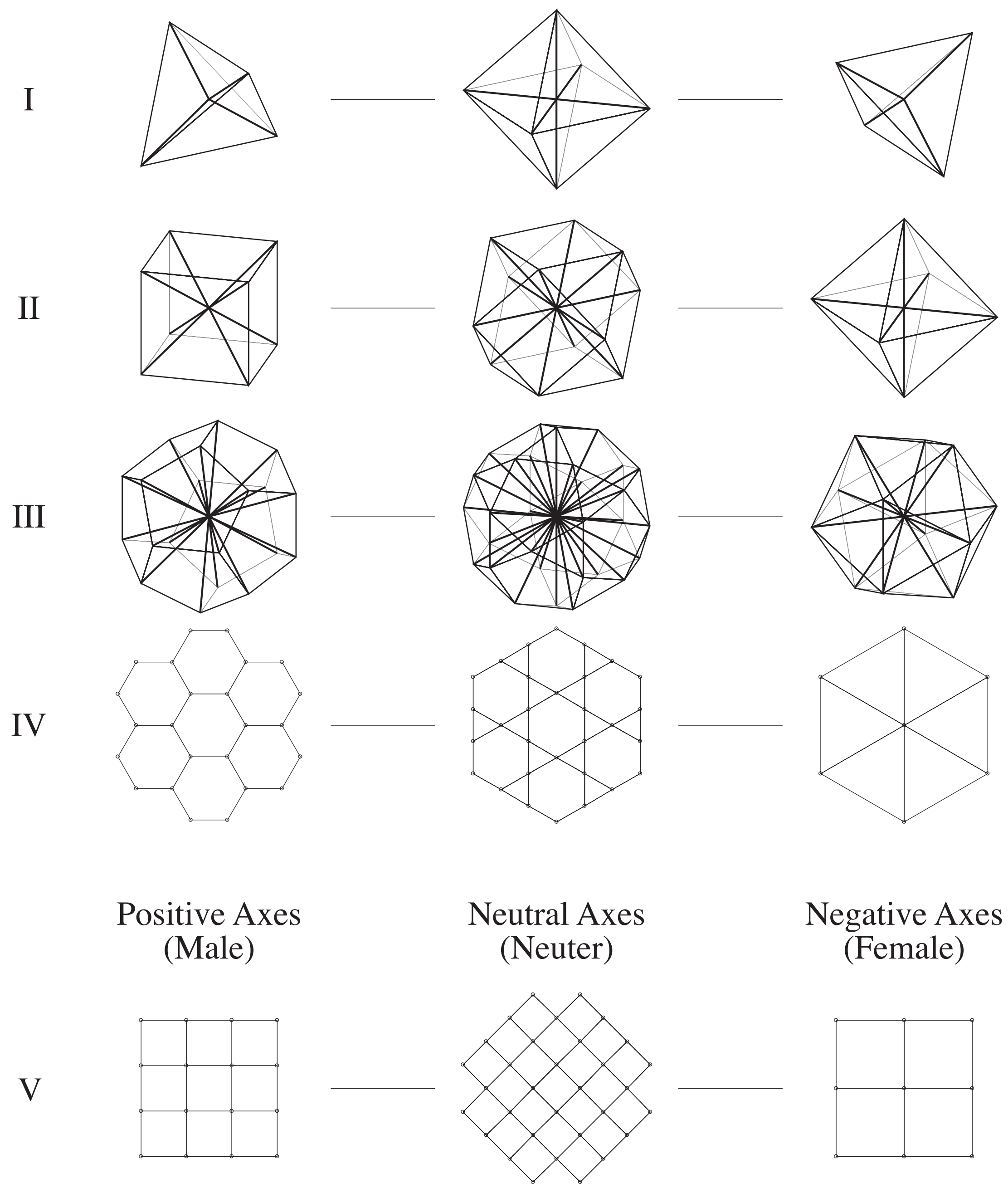


Figure 2 : Symmetry systems of positive, neutral and negative axes of the five classes. Axes through the vertices of the elements shown pass through the centres of the polyhedra in Classes I-III, and are orthogonal to the page in Classes IV and V.

SYSTEMS OF POSITIVE, NEUTRAL AND NEGATIVE AXES OF EACH CLASS

Systems of positive and negative axes are dual for each of the five classes. Class V is akin to Class I, in that its poles are self-dual.

The positive axes of each symmetry system are given by the centres of one of the two sets of faces of the central elements of first degree, as shown in the left column of Fig 2.

Class I	Class II	Class III	Class IV	Class V
centres of the triangular faces of the TetraTetrahedron *(one tetrahedral set only)*	centres of the triangular faces of the OctaHexahedron	centres of the triangular faces of the IcosiDodecahedron	centres of the triangular faces of the TriHexagonal Array	centres of the square faces of the SquareSquare Array *(one square set only)*

Table 2: Positive Axes

In the polyhedral classes, these accord with the axes of the centres of the faces of the Male Tetrahedron, Octahedron, and Icosahedron; and of the vertices of the Female Tetrahedron, Hexahedron, and Dodecahedron.

In Class IV, these accord with the axes of the centres of the faces of the Triangular Array, and of the vertices of the Hexagonal array; and form a hexagonal grid.

In Class V, these accord with the axes of the centres of the faces of the Male Square Array, and of the vertices of the Female Square Array; and form a square grid.

The neutral axes of each symmetry system are given by the vertices of the central elements of first degree, as shown in the middle column of Fig 2 opposite.

Class I	Class II	Class III	Class IV	Class V
vertices of the TetraTetrahedron	vertices of the OctaHexahedron	vertices of the IcosiDodecahedron	vertices of the TriHexagonal Array	vertices of the SquareSquare Array

Table 3: Neutral Axes

These correspond to the axes of the mid-edges of the male and female polar elements, i.e. Male and Female Tetrahedra, Octahedron and Hexahedron, Icosahedron and Dodecahedron, Triangular and Hexagonal Array, and Male and Female Square Arrays. The neutral axes of Class IV form a trihexagonal grid; those of Class V form a square grid on the √2 diagonal.

The negative axes of each system are given by the centres of the other of their two sets of faces of the central elements:

Class I	Class II	Class III	Class IV	Class V
centres of the triangular faces of the TetraTetrahedron *(other tetrahedral set only)*	centres of the square faces of the OctaHexahedron	centres of the pentagonal faces of the IcosiDodecahedron	centres of the hexagonal faces of the TriHexagonal Array	centres of the square faces of the SquareSquare Array *(other square set only)*

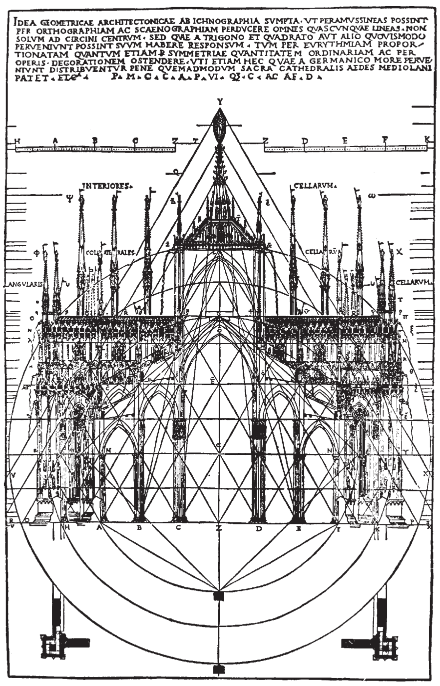
Table 4: Negative Axes

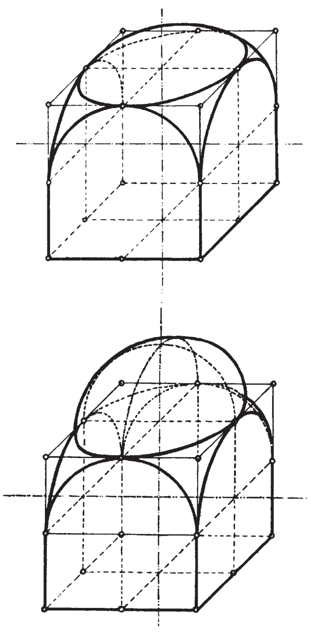
These accord with the axes of the vertices of the Male Tetrahedron, Octahedron, Icosahedron, Triangular Array, and Male Square Array as shown in the right column of Fig 2; and with the axes of the centres of the faces of the Female Tetrahedron, Hexahedron, Dodecahedron, Hexagonal Array, and Female Square Array. The negative axes of Class IV form a triangular grid, while those of Class V form a square grid.

In the first three classes, with certain regular exceptions, *each face of every regular or semi-regular polyhedra is orthogonal to and centred about its respective fundamental axis*, that is the specific axis of the particular symmetry pattern with which it is associated. The exceptions are the paired neutral faces of the three Snub Polyhedra, there then being two adjoining neutral faces for each neutral axis instead of one, with their common edge being orthogonal to and centred about the axis.[12] Similarly in the last two classes, with certain regular exceptions, *each face of every regular or semi-regular tiling is centred about its respective fundamental axis*. The exceptions are the paired neutral faces of the two Skew Polygonal Tilings, there being two adjoining neutral faces for each neutral axis instead of one, with their common edge being bisected by the axis.

Note that the positive and negative axes of Class I taken together are the positive axes of Class II, which are in turn a subset of the positive axes of Class III. The neutral axes of Class I are the negative axes of Class II and a subset of the neutral axes of Class III. Positive, neutral and negative axes of Class V are square grids, the neutral axis being rotated by $\pi/4$ to the other two and scaled.

The shaded illustrations of polyhedra and tilings consistently depict male faces as dark, neutral faces as medium, and female faces as light shading. Within each illustration, the circumradii of polyhedra depicted are constant. It is of course only possible to illustrate a representative portion of each element in Classes IV and V, these elements being of infinite extent. All three-dimensional views are either projections of constant rotation of the regulating cube,[13] or lie along female axes.[14] Classes IV and V also are shown centred about a female axis. It is generally more convenient to depict Class V elements in line with elements of the other four classes, but their proper relationship is orthogonal to that axis from the corresponding Class II element.





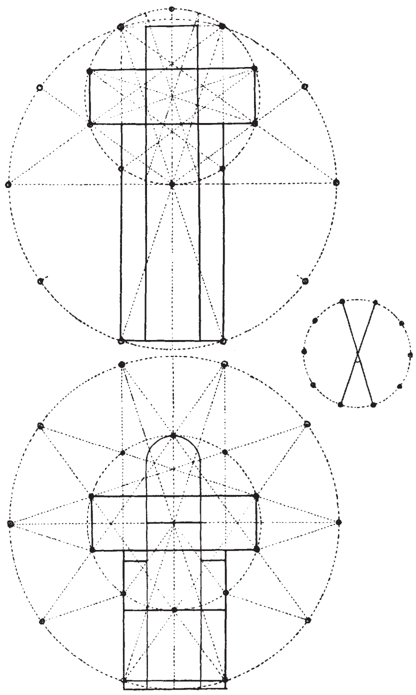


Figure 3 : Ad Triangulum, Ad Quadratum, and Ad Pentangulum proportional harmonies used in traditional architecture (after Ghyka). Left: Dome of Milan, Caesar Caesariano, 1521; centre: Cuboctahedron and Byzantine Cupolas; right: Two Gothic Standard Plans (Moessel).

THE THREE PROPORTIONAL SYSTEMS INHERENT IN THE SYMMETRY AXES

The KAIROS School teaches three traditional proportional systems which have been used in sacred architecture throughout history.[15] Fig 3, after Ghyka,[16] shows examples of each of these. They are:

- Ad Triangulum, which exploits √3 ratios derived from the equilateral triangle;
- Ad Quadratum, which exploits √2 ratios derived from the square;[17] and
- Ad Pentangulum, which exploits $ø = {}^{(\sqrt{5}+1)}/_2$ (Golden Section) ratios, derived from the regular pentagon, and from the √5 diagonal of the double square.

Positive axes of Classes I-IV develop threefold symmetry, with Class V of fourfold symmetry.

Neutral axes of all five classes develop twofold symmetry, including the neutral square faces of the rhombic solids.

Negative axes develop threefold, fourfold, fivefold, sixfold, and fourfold symmetry respectively. In the case of the snub and skew elements, the symmetries are rotational only but not in general reflective.[18] It will then be seen that the three polyhedral classes exhibit Ad Triangulum, Ad Quadratum and Ad Pentangulum proportional harmonies respectively, interwoven with Ad Triangulum. Class IV is of Ad Triangulum harmony, and Class V of Ad Quadratum. A limited Ad Quadratum is developed throughout all classes about the neutral axes.

	POSITIVE	NEUTRAL	NEGATIVE	SYMMETRY	KEY PROPORTION	NUMBER	POLYGON
I	3	2	3	{2,3,3}	√3 Ad Triangulum (√3)	3	Triangle
II	3	2	4	{2,3,4}	√4 Ad Quadratum (√2)	4	Square
III	3	2	5	{2,3,5}	√5 Ad Pentangulum (ø)	5	Pentagon
IV	3	2	6	{2,3,6}	√6 Ad Triangulum (√3)	6	Hexagon
V	4	2	4	{2,4,4}	√4 Ad Quadratum (√2)	4	Square

THE POLAR ELEMENTS AND SHORT HORIZONTAL SEQUENCES OF FIRST DEGREE

The five Platonic polyhedra are reorganised as six polar solids comprising three male-female duals.[19] The three Platonic tilings are reorganised as four polar tilings comprising two male-female duals, of triangular and hexagonal grid, and square and square grid respectively. Fig 4 shows how these polarise about the central neutral element as one pair for each set.

Thus the three neutral polyhedra (tetratetrahedron, octahexahedron and icosidodecahedron) and two neutral tilings (triangularhexagonal and squaresquare arrays) are recognised as mediating states of their respective polarities. Quasi-regular neutral central and regular polar elements of each set are the main elements of their respective horizontal sequences.

	(Regular) MALE POLAR		(Quasi-Regular) NEUTRAL CENTRAL		(Regular) FEMALE POLAR
I	♂ Tetrahedron	————————	TetraTetrahedron	————————	♀ Tetrahedron
II	Octahedron	————————	OctaHexahedron	————————	Hexahedron
III	Icosahedron	————————	IcosiDodecahedron	————————	Dodecahedron
IV	Triangular Array	————————	TriHexagonal Array	————————	Hexagonal Array
V	♂ Square Array	————————	SquareSquare Array	————————	♀ Square Array

THE INTERMEDIARY ELEMENTS AND LONG HORIZONTAL SEQUENCES OF FIRST DEGREE

The long horizontal sequences shown in Figs 5 and 12 are completed by the inclusion of two intermediary elements for each set. The octahedron-cube and icosahedron-dodecahedron truncation sequences shown are well-known, and are illustrated by Holden.[20] The intermediary solids are the truncations of their respective polar solids, as the intermediary tilings are the truncations of their respective polar tilings.

The intermediary elements of first degree of Class IV are the hexagonal-hexagonal grid which is developed as a truncation of the male polar triangular grid, and the triangular-dodecagonal grid which is developed as a truncation of the female polar hexagonal grid. The intermediary elements of first degree of Class V are the octagonal-square grid developed as a truncation of the male polar square grid, and the square-octagonal grid which is developed as a truncation of the female polar square grid.

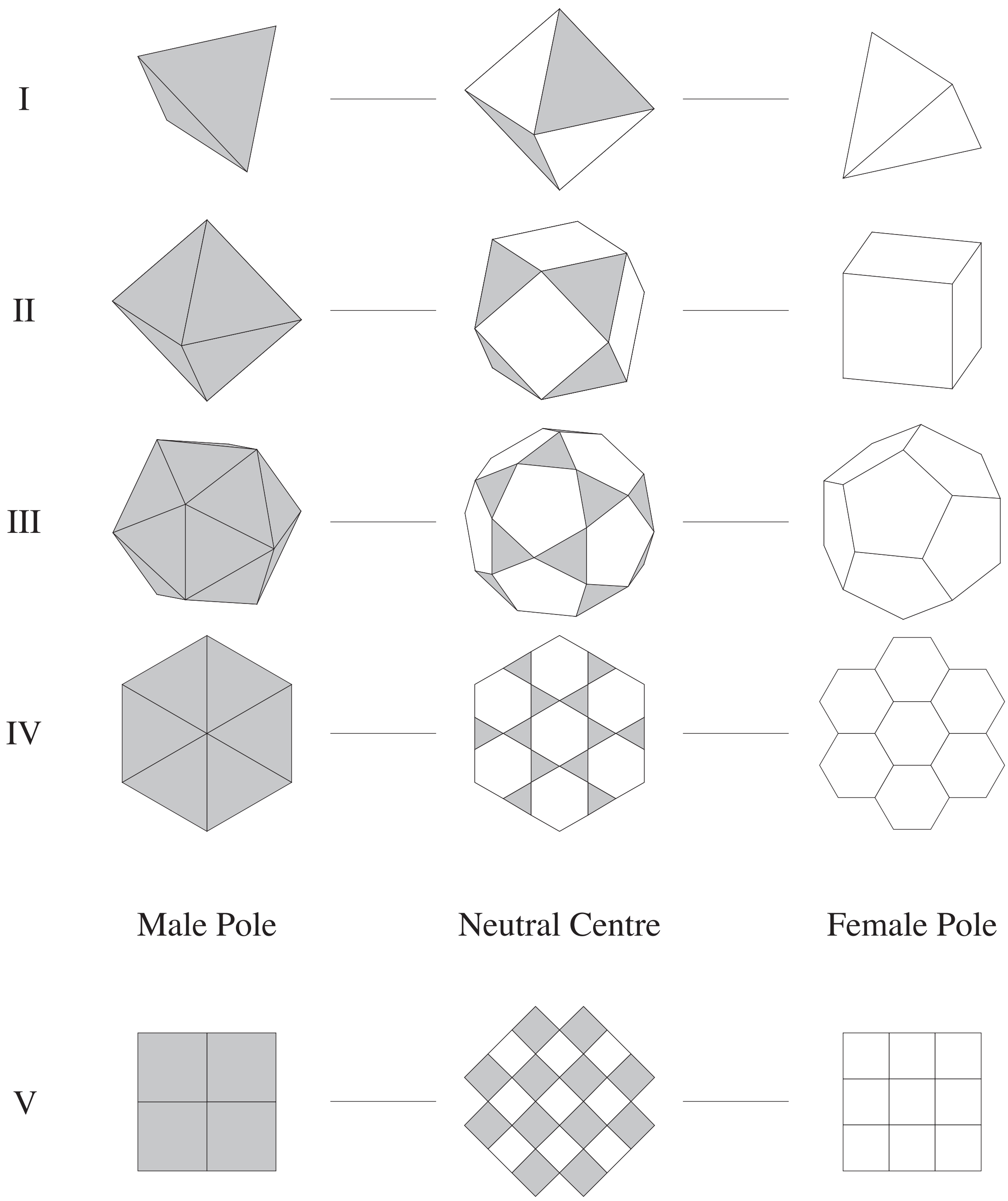


Figure 4 : Regular polyhedra and tilings polarised in duals about the quasi-regular polyhedra and tilings.

	MALE ♂ POLAR	MALE INTERMEDIARY	NEUTRAL CENTRAL	FEMALE INTERMEDIARY	FEMALE ♀ POLAR
I	Male ♂ Tetrahedron	Truncated Male Tetrahedron	*(Octahedron)* TetraTetrahedron	Truncated Female Tetrahedron	Female ♀ Tetrahedron
II	Octahedron	Truncated OctaHexahedron	*(Cuboctahedron)* Hexahedron	Truncated Hexahedron	*(Cube)* Hexahedron
III	Icosahedron	Truncated Icosahedron	IcosiDodecahedron	Truncated Dodecahedron	*(Pentagonal-)* Dodecahedron
IV	Triangular Array	Truncated Triangular Array	TriHexagonal Array	Truncated Hexagonal Array	Hexagonal Array
V	Male ♂ Square Array	Truncated Male Square Array	SquareSquare Array	Truncated Female Square Array	Female ♀ Square Array

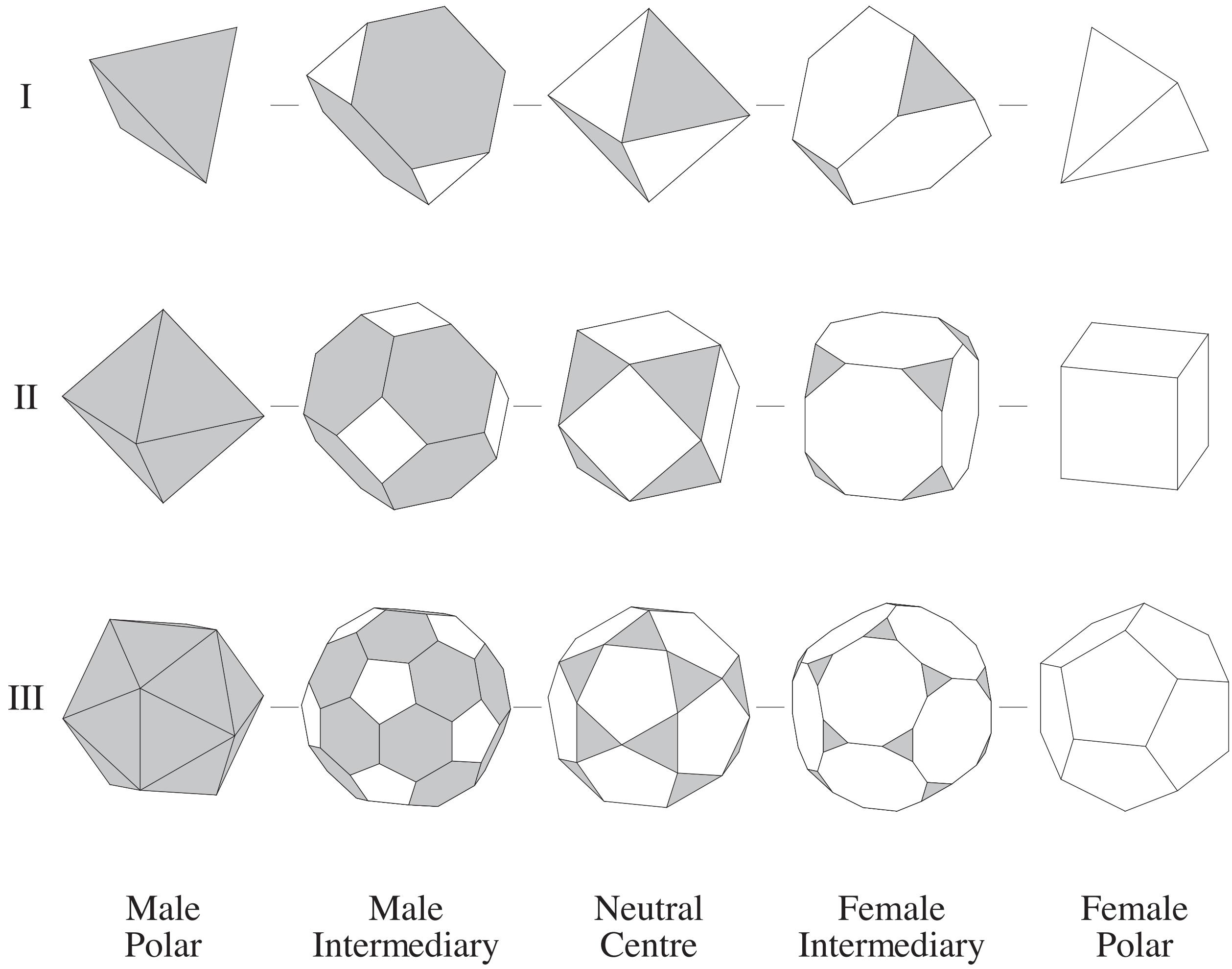


Figure 5 : Long Horizontal Sequences of Classes I-III showing the regular and semi-regular polyhedra of First Degree. See also Fig 12.

THE NEUTRAL CENTRAL ELEMENTS OF SECOND DEGREE

To each of the three central elements of first degree, there exist corresponding central elements of second and of third degree. They are shown as the second row of Figs 6 and 13. Each second degree central solid is the small rhombic solid of the respective solid of first degree: the Small Rhombic TetraTetrahedron, the Small Rhombic OctaHexahedron, and the Small Rhombic IcosiDodecahedron. These solids are developed from their first degree solids with all faces displacing regularly outwards along their axes, (or with faces contracting symmetrically), while:

- rotating the positive faces about their positive axes by half their central face angle, (triangular faces of Classes I, II, and II by $^{\pi}/3$);
- rotating the negative faces about their negative axes by half their central angle, (triangular faces of Class I by $^{\pi}/3$, square faces of Class II by $^{\pi}/4$, pentagonal faces of Class III by $^{\pi}/5$); and
- developing square faces on the neutral axes to complete the solids.

The second degree central tilings are the small rhombic tiling of the respective tiling of first degree: the triangular-square-hexagonal grid, and the square-square-square grid, or the Small Rhombic TriHexagonal Array and the Small Rhombic SquareSquare Array. These tilings are shown in the second row of Fig 13, and are developed from their first degree tilings with all axes expanding regularly from one another, (or with faces contracting symmetrically), while:

- rotating the positive faces about their positive axes by half their central face angle, (triangular faces of Class IV by $^{\pi}/3$ and square faces of Class V by $^{\pi}/4$);
- rotating the negative faces about their negative axes by half their central angle, (hexagonal faces of Class IV by $^{\pi}/6$ and square faces of Class V by $^{\pi}/4$); and
- developing square faces on the neutral axes to complete the tilings.

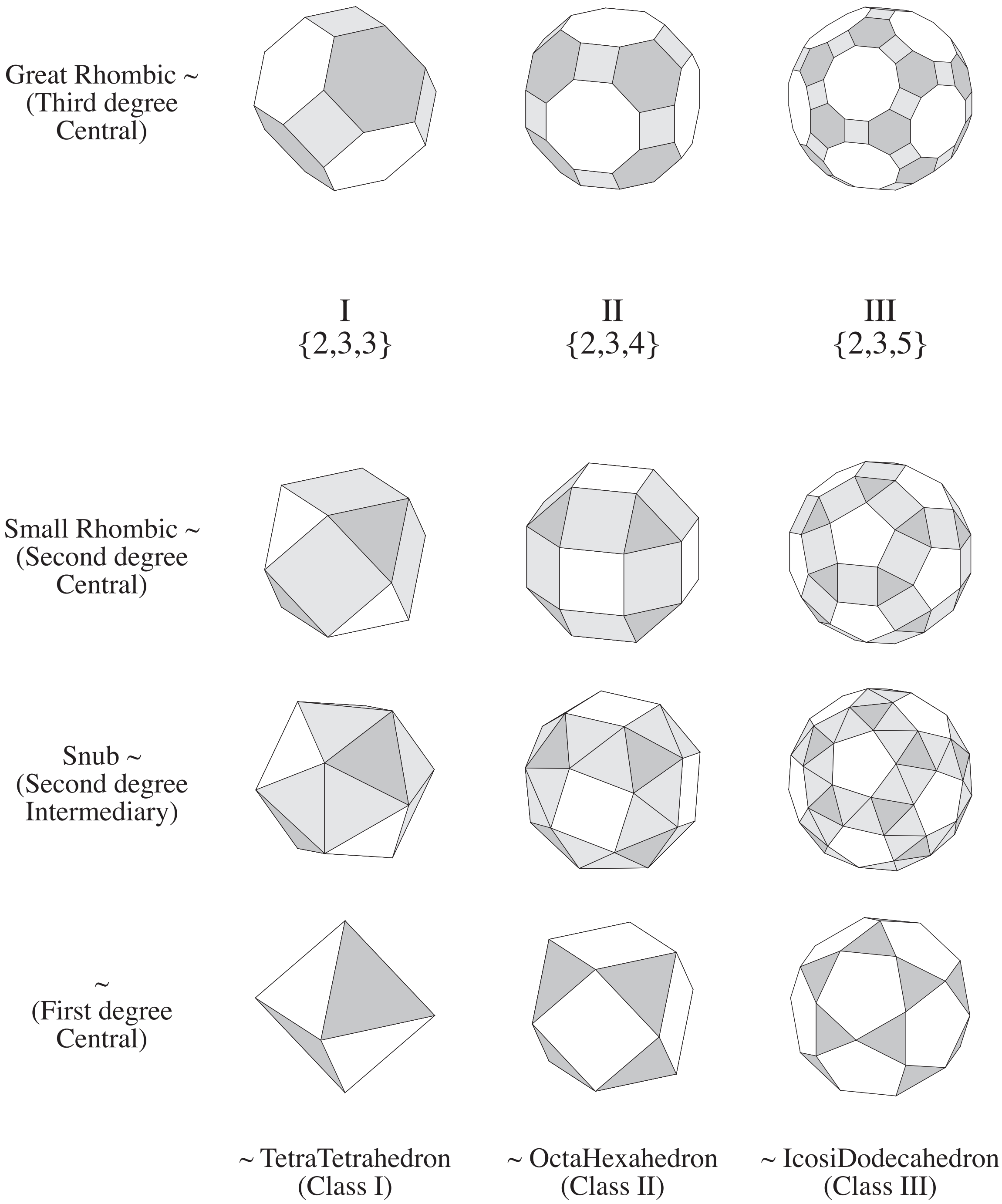


Figure 6 : Vertical Sequences of Classes I-III showing the Neutral Central polyhedra of First, Second and Third degree together with the Intermediary polyhedra of Second degree. See also Fig 13.

THE INTERMEDIARY
ELEMENTS OF SECOND DEGREE

Part way through the rotational displacements to obtain the central elements of second degree, the Snub or Skew elements are formed as intermediary elements of second degree. These are shown as the third row of Figs 6 and 13. For convenience, I illustrate only right-hand forms, which represent both enantiomorphs for each case. Of particular interest is the recognition of the true Snub TetraTetrahedron of Class I as the regular Icosahedron (of Class III), considered in its alternative mode.[21] This Snub TetraTetrahedron of Class I is strictly paralleled by the Snub OctaHexahedron of Class II and the Snub IcosiDodecahedron of Class III. In Class IV and V, the snub intermediary elements of second degree are given by the skew triangular-triangular-hexagonal grid and the skew square-diamond-square grid, or Skew TriHexagonal and Skew SquareSquare Arrays.

These rotational displacement sequences are the precise equivalents of R. Buckminster Fuller's jitterbug systems. He gives the (Class I) symmetrical contraction of the vector equilibrium (small rhombic tetra tetrahedron) through the icosahedron (snub tetra tetrahedron) to the octahedron (tetra tetrahedron).[22] As far as I am aware, neither Fuller nor his followers discovered the equivalent Class II and III jitterbugs,[23] i.e. the symmetrical contraction from small rhombic octahexahedron through snub octahexahedron to octahexahedron, and that from small rhombic icosidodecahedron to snub icosidodecahedron to icosidodecahedron. Rigid male triangles alternate with rigid female squares and pentagons respectively to which they are vertex pin-jointed, with the neutral faces being voids. I have deduced these from the order I advance, and the predictability of these behaviours across the classes is further confirmation of its validity.

Analogous two-dimensional jitterbug systems also exist for Classes IV and V, with rigid male triangles and squares alternating with female hexagons and squares to which they are vertex pin-jointed, and with neutral faces being voids. In Class IV, the small rhombic trihex symmetrically contracts through the skew trihex to the trihex. In Class V, the small rhombic squaresquare symmetrically contracts through the skew squaresquare to the squaresquare grid.

THE NEUTRAL CENTRAL ELEMENTS OF THIRD DEGREE

Just as the neutral central elements of second degree can be thought of as higher octaves of the corresponding elements of first degree, as corresponding faces rotate and displace from each other, the neutral central elements of third degree represent a higher octave again, as corresponding faces double in frequency and displace from each other. This is why the pattern is given vertical articulation. They are shown as the first row of Figs 6 and 13. Within each class, the element of third degree is the greater element of second degree, and the truncated element of first degree. The truncated tetra tetrahedron or truncated octahedron is known as the Great Rhombic TetraTetrahedron, the truncated octahexahedron or truncated cuboctahedron as the Great Rhombic OctaHexahedron, and the truncated icosidodecahedron as the Great Rhombic IcosiDodecahedron. Referring to the last two solids, Cundy and Rollett suggest that the "great rhombic" terminology is preferable, as the truncations require an additional distortion to convert rectangles to squares.[24] The syllable 'rhomb-' shows that the neutral sets of square faces lie in the planes of the rhombic hexahedron (cube), rhombic dodecahedron, and rhombic triacontahedron respectively, which are the duals of the central TetraTetrahedron, OctaHexahedron and IcosiDodecahedron. These three duals are illustrated in Fig 14 below. The central elements of third degree of Classes IV and V are given by the hexagonal-square-dodecagonal grid and the octagonal-square-octagonal grid, or Great Rhombic TriHexagonal and Great Rhombic SquareSquare Arrays. Their duals, the diamond and square arrays, are known as the rhombic trihexagonal and rhombic squaresquare arrays, also shown in Fig 14.

CLASS Symmetry	I {2,3,3}	II {2,3,4}	III {2,3,5}	IV {2,3,6}	V {2,4,4}
Central Element of 3rd Degree	Great Rhombic TetraTetrahedron	Great Rhombic OctaHexahedron	Great Rhombic IcosiDodecahedron	Great Rhombic TriHexagon Array	Great Rhombic SquareSquareArray
(Inter. Element of 3rd Degree)	——	——	——	——	——
Central Element of 2nd Degree	Small Rhombic TetraTetrahedron	Small Rhombic OctaHexahedron	Small Rhombic IcosiDodecahedron	Small Rhombic TriHexagon Array	Small Rhombic SquareSquareArray
Inter. Element of 2nd Degree	Snub TetraTetrahedron	Snub OctaHexahedron	Snub IcosiDodecahedron	Skew TriHexagon Array	Skew SquareSquareArray
Central Element of 1st Degree	*(Octahedron)* TetraTetrahedron	*(Cuboctahedron)* OctaHexahedron	IcosiDodecahedron	TriHexagon Array	SquareSquareArray

So for the elements of third degree:

- Positive faces are formed of double frequency on the positive axes (hexagonal for Classes I-IV, octagonal for Class V);
- Square Neutral faces are developed on the neutral axes of all five Great Rhombs; and
- Negative faces are formed of double frequency on the negative axes (hexagonal for the Great Rhombic TetraTetrahedron, octagonal for the Great Rhombic OctaHexahedron, decagonal for the Great Rhombic IcosiDodecahedron, dodecagonal for the Great Rhombic TriHexagonal Array, and octagonal for the Great Rhombic SquareSquare Array).

Clearly, for each class, Neutral Central, Small Rhombic and Great Rhombic elements form a major progression, whilst Neutral Central, Snub/Skew and Small Rhombic form a minor progression. All four are obviously central elements, rather than polar or intermediary polar.

There are no intermediary elements of third degree. But it will be appreciated that the linear sequential order of the vertical axis is less definitive than the horizontal.

THE META-PATTERN OF EACH SET

So we have discerned a structural pattern of aspects for each set that accords well with traditional symbolism. In its short form, this takes the form of an inverted T, shown in Table 6. It comprises a central trunk of the central elements of first, second, and third degree, with the polar elements of first degree either side. This is an intelligible and imageable structure, that is readily memorised and is useful for teaching.

In its expanded form, the structure takes the form of an "i". Figs 7-11 show this structure for each of the five classes. The intermediary cases are included - there being no intermediary between the third degree and second degree neutral centres. I develop Grünbaum and Shephard's tiling and colouring symbols for the polyhedra and tilings. The first numerical sequence gives the sequence of frequencies of faces about a vertex, commencing always from a female axis and proceeding clockwise, and differentiating between male, female and neutral faces. The second expression in brackets gives the colouring of faces in the same sequence, and therefore the axis type, 1 being male, 2 female, and 3 neutral. Symbolic description & colouring of an element are consistent throughout the classes.

Although Class IV and V generate all but one of the eleven regular and semi-regular tilings, they by no means exhaust all thirty-two *uniform colourings* of these tilings, which Grünbaum and Shephard illustrate in Fig 2.9.2.[25] (Note that (3^6) 121314 appears to be incorrectly shaded in that illustration). Colouring the tilings modifies the symmetry system in a number of cases. Two coloured tilings although of the correct {2,4,4}-fold symmetry do not appear in this classification. For the remainder, they either belong to the degenerate {2,2,2,2}-fold symmetry class, or to the {3,3,3}-fold symmetry class, both of which I exclude. Although it might be thought possible to extend the order to include classes of the remaining {2,4,4}-fold, and {3,3,3}-fold and {2,2,2,2}-fold symmetry elements to accommodate all of the uniform colourings, these classes even if they exist are less regularly defined and incomplete.[26]

Regular and semi-regular elements that repeat, shown in Table 7, are discerned by disregarding the differences between them in colouring and corresponding ascription of faces to symmetry axes. Six polyhedra appear twice in the first three classes. In Class IV, one tiling appears twice; in Class V, one tiling appears four times, and another appears three times.

A META-PATTERN FOR THE THREE CLASSES OF POLYHEDRA

The arrangement for each polyhedral set suggests a convenient overall pattern of the first three classes inside a cubic matrix. A 3x3x3 cube contains the three short sets as vertical layers. A 5x5x5 cube contains the complete sets in the outside and central vertical layers, with two empty inbetween layers.

With certain regular exceptions, each axis has its characteristic frequency of rotational and reflective symmetry. Each face of every polyhedron is orthogonal to and centred about its respective fundamental axis, i.e. the specific axis of the particular symmetry pattern of its class.

The only exceptions arise in the Snub polyhedra. Here the symmetries are rotational but not reflective. Positive and negative faces remain orthogonal to their respective axes, but are displaced along their axes from the central solids of first degree and rotated by an irrational measure. But on the neutral axes, pairs of triangular faces form about a common edge that is perpendicular to and bisected by their axis. These neutral faces are not perpendicular to the neutral axes, nor do the axes pass through the midpoints of the faces.

	I	II	III	IV
CENTRE	TetraTetrahedron	OctaHexahedron	IcosiDodecahedron	TriHexagon Array
RHOMBIC DUAL	Dual Rhombic TetraTetrahedron *(Rhombic Hexahedron)*	Dual Rhombic OctaHexahedron *(Rhombic Dodecahedron)*	Dual Rhombic IcosiDodecahedron *(Rhombic Triacontahedron)*	Dual Rhombic TriHexagon Array *(Diamond Array)*

	V
CENTRE	SquareSquareArray
RHOMBIC DUAL	Dual Rhombic SquareSquareArray *(Square Array)*

Table 5: Rhombic Duals

	3°				GREAT RHOMB	
	2°				SMALL RHOMB	
♂	1°	♀		MALE POLE	NEUTRAL CORE	FEMALE POLE

Table 6: Meta-Pattern of each set

I	Male Tetrahedron	3^3 (111)	=	3^3 (222)	Female Tetrahedron	I
I	Male Truncated Tetrahedron	3.6^2 (211)	=	$6^2.3$ (221)	Truncated Female Tetrahedron	I
I	TetraTetrahedron	3.3.3.3 (2121)	=	3^4 (1111)	Octahedron	II
I	Snub TetraTetrahedron	$3.3.3.3^2$ (23133)	=	3^5 (11111)	Icosahedron	III
I	Small Rhombic TetraTetrahedron	3.4.3.4 (2313)	=	4.3.4.3 (2121)	OctaHexahedron	II
I	Great Rhombic TetraTetrahedron	6.4.6 (231)	=	4.6^2 (211)	Truncated Octahedron	II
IV	Truncated Triangular Array	6.6^2 (2121)	=	6^3 (222)	Hexagonal Array	IV
V	Male Square Array	4^4 (1111)	=	4^4 (2222)	Female Square Array	V
			=	4.4.4.4 (2121)	Central SquareSquare Array	V
			=	4.4.4.4 (2313)	Small Rhombic SquareSquare	V
V	Male Truncated Square Array	4.8^2 (211)	=	$8^2.4$ (221)	Female Truncated Square Array	V
			=	8.4.8	Great Rhombic SquareSquare	V

Table 7: Regular and semi-regular elements that repeat
(disregarding alternative symmetry patterns given by alternative colourings of faces)

META-PATTERN: {2, m, n}

Great Rhombic Neutral Element 2n.4.2m (231) *Third Degree* *Central*

(m = 3, n = 3, 4, 5, 6; m = 4, n = 4)

See Figs 7-11.

Small Rhombic Neutral Element n.4.m.4 (2313) *Second Degree* *Central*
Snub Neutral Element $n.3.m.3^2$ (23133) *Second Degree* *Intermediary*

Male Polar Element m^n (1^n) *Male* *Polar*	Truncate Male Polar Element n.2m.2m (211) *Male* *Intermediary*	Central Neutral Element n.m.n.m (2121) *First Degree* *Central*	Truncate Female Polar Element 2n.2n.m (221) *Female* *Intermediary*	Female Polar Element n^m (2^m) *Female* *Polar*

CLASS I: {2, 3, 3}

Great Rhombic TetraTetrahedron 6.4.6 (231) *Truncated* *Octahedron*

TETRATETRAHEDRAL

See Fig. 7.

Small Rhombic TetraTetrahedron 3.4.3.4 (2313) *Cuboctahedron*
Snub TetraTetrahedron $3.3.3.3^2$ (23133) *Icosahedron*

Male Tetrahedron 3^3 (111) *Tetrahedron*	Truncate Male Tetrahedron 3.6^2 (211) *Truncated* *Tetrahedron*	Central TetraTetrahedron 3.3.3.3 (2121) *Octahedron*	Truncate Female Tetrahedron $6^2.3$ (221) *Truncated* *Tetrahedron*	Female Tetrahedron 3^3 (222) *Tetrahedron*

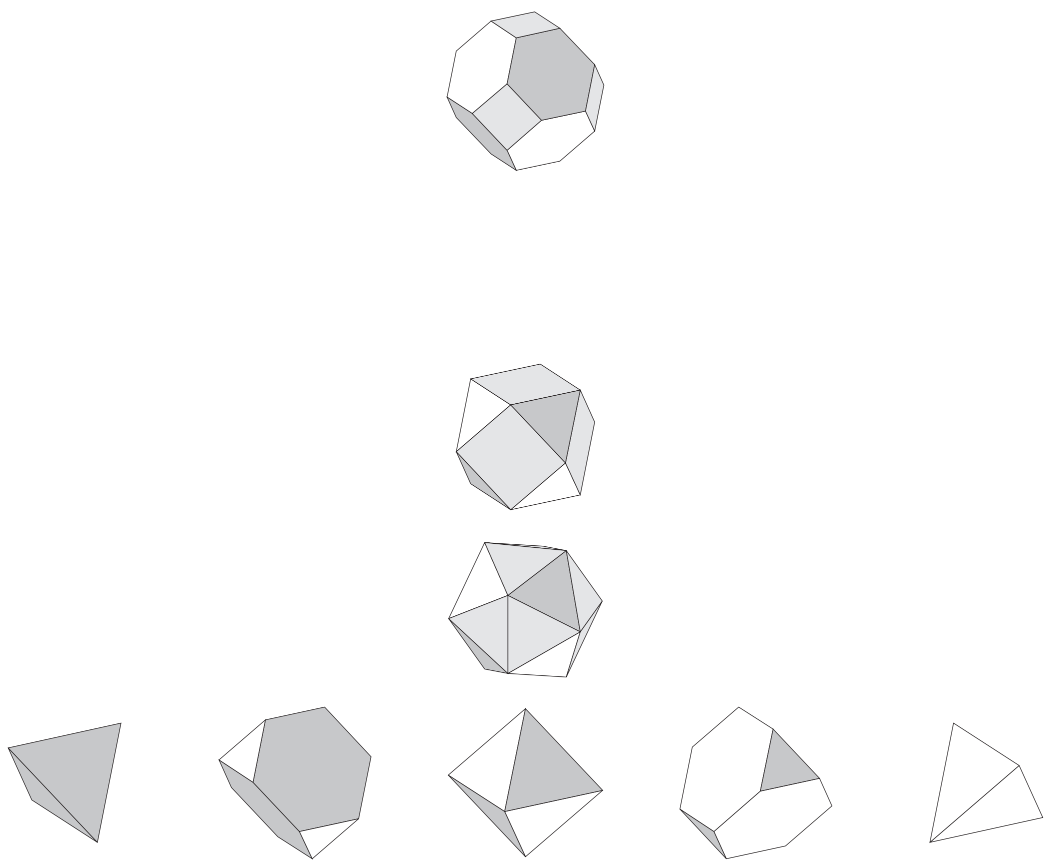


Figure 7 : Regular and semi-regular polyhedra of Class I of TetraTetrahedral symmetry.

CLASS II: {2, 3, 4}

Great Rhombic
OctaHexahedron
8.4.6
(231)
Truncated
OctaHexahedron

OCTAHEXAHEDRAL

See Fig. 8.

Small Rhombic
OctaHexahedron
4.4.3.4
(2313)
"Square Spin"

Snub
OctaHexahedron
$4.3.3.3^2$
(23133)
Snub
Cuboctahedron

Male Octahedron 3^4 (1111)	Truncate Male Octahedron 4.6^2 (211) *"Mecon"*	Central OctaHexahedron 4.3.4.3 (2121) *Cuboctahedron* *or "Dymaxion"*	Truncate Female Hexahedron $8^2.3$ (221) *Truncated Cube*	Female Hexahedron 4^3 (222) *Cube*

CLASS III: {2, 3, 5}

Great Rhombic
IcosiDodecahedron
10.4.6
(231)
Truncated
Icosidodecahdrn

ICOSIDODECAHEDRAL

See Fig. 9.

Small Rhombic
IcosiDodecahedron
5.4.3.4
(2313)
Rhombic
Dodecahedron

Snub
IcosiDodecahedron
$5.3.3.3^2$
(23133)
Snub
Dodecahedron

Icosahedron 3^5 (11111)	Truncate Male Icosahedron 5.6^2 (211)	IcosiDodecahedron 5.3.5.3 (2121) *First*	Truncate Female Dodecahedron $10^2.3$ (221)	Dodecahedron 5^3 (222) *Pentagonal* *Dodecahedron*

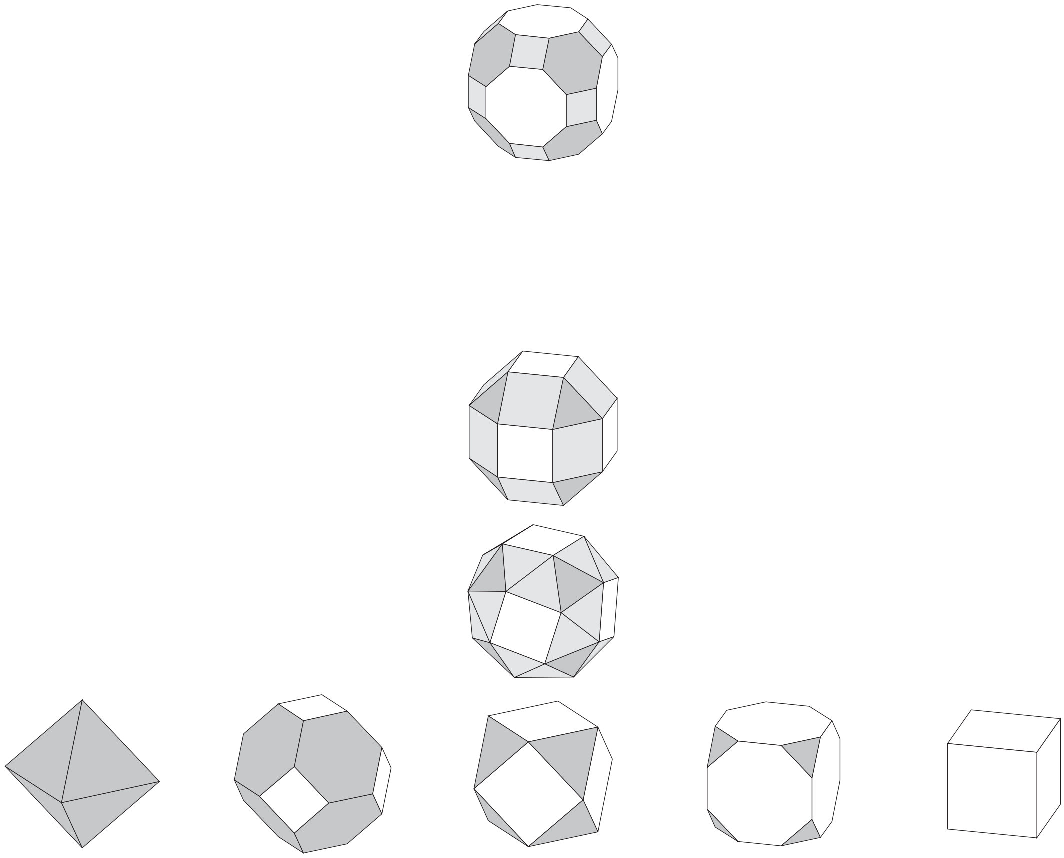


Figure 8 : Regular and semi-regular polyhedra of Class II of OctaHexahedral symmetry.

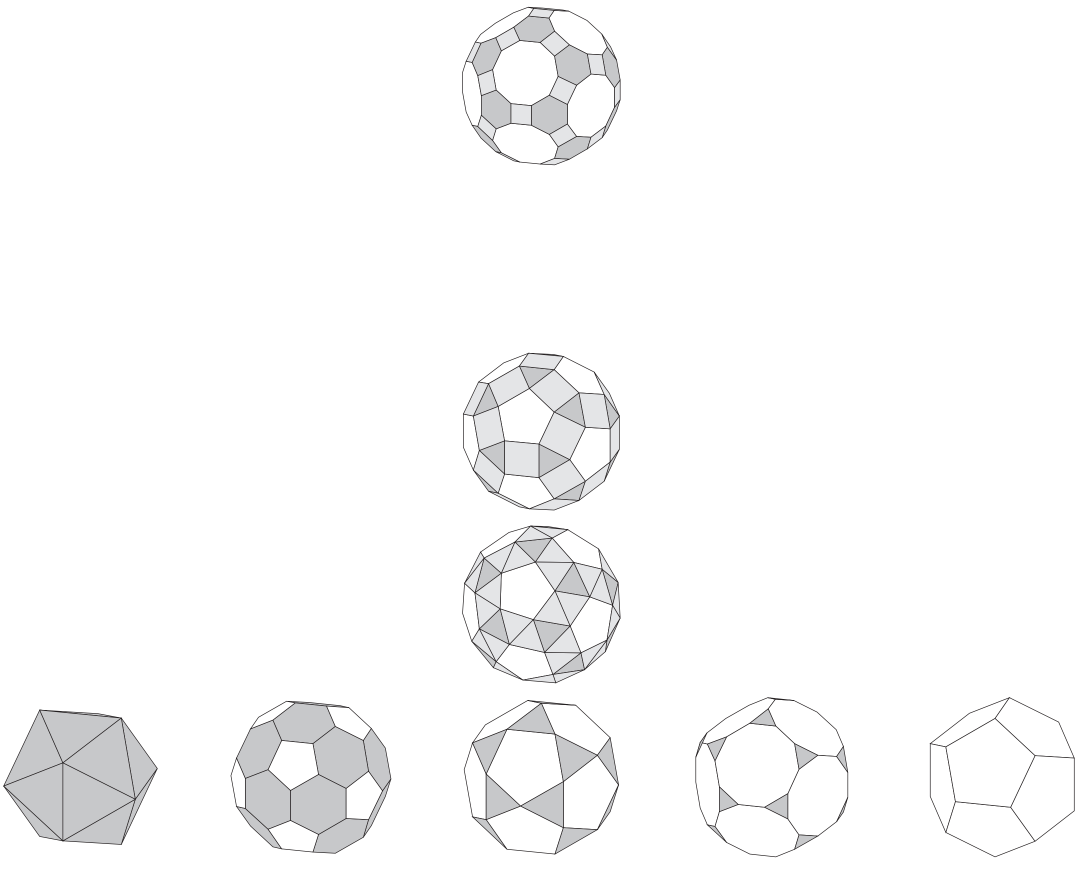


Figure 9 : Regular and semi-regular polyhedra of Class III of IcosiDodecahedral Symmetry.

CLASS IV: {2, 3, 6}

TRIANGULAR-HEXAGONAL

See Fig 10.

Great Rhombic
TriHex Array
12.4.6
(231)
Third Degree
Central

Small Rhombic
TriHex Array
6.4.3.4
(2313)
Second Degree
Central

Skew
TriHex Array
$6.3.3.3^2$
(23133)
Second Degree
Intermediary

Triangular Array 3^6 (111111) *Triangular Grid*	Truncate Triangular Array 6.6^2 (211) *Hexagonal Array*	TriHex Array 6.3.6.3 (2121) *Triangular-* *Hexagonal Array*	Truncate Hexagonal Array $12^2.3$ (221) *Intermediary*	Female Hexagonal Array 6^3 (222) *Tri-Dodecagonal* *Hexagonal Grid*

CLASS V: {2, 4, 4}

SQUARE-SQUARE

See Fig. 11.

Great Rhombic
SqrSquare Array
8.4.8
(231)
Third Degree
Central

Small Rhombic
SqrSquare Array
4.4.4.4
(2313)
Second Degree
Central

Skew
SqrSquare Array
$4.3.4.3^2$
(23133)
Second Degree
Intermediary

Male Square Array 4^4 (1111) *Square Grid*	Truncate Male Square Array 4.8^2 (211) *Octangular-* *Square Array*	SquareSquare Array 4.4.4.4 (2121) *Chess Board*	Truncate Female Square Array $8^2.4$ (221) *Square-* *Octangular Array*	Female Square Array 4^4 (2222) *Square Grid*

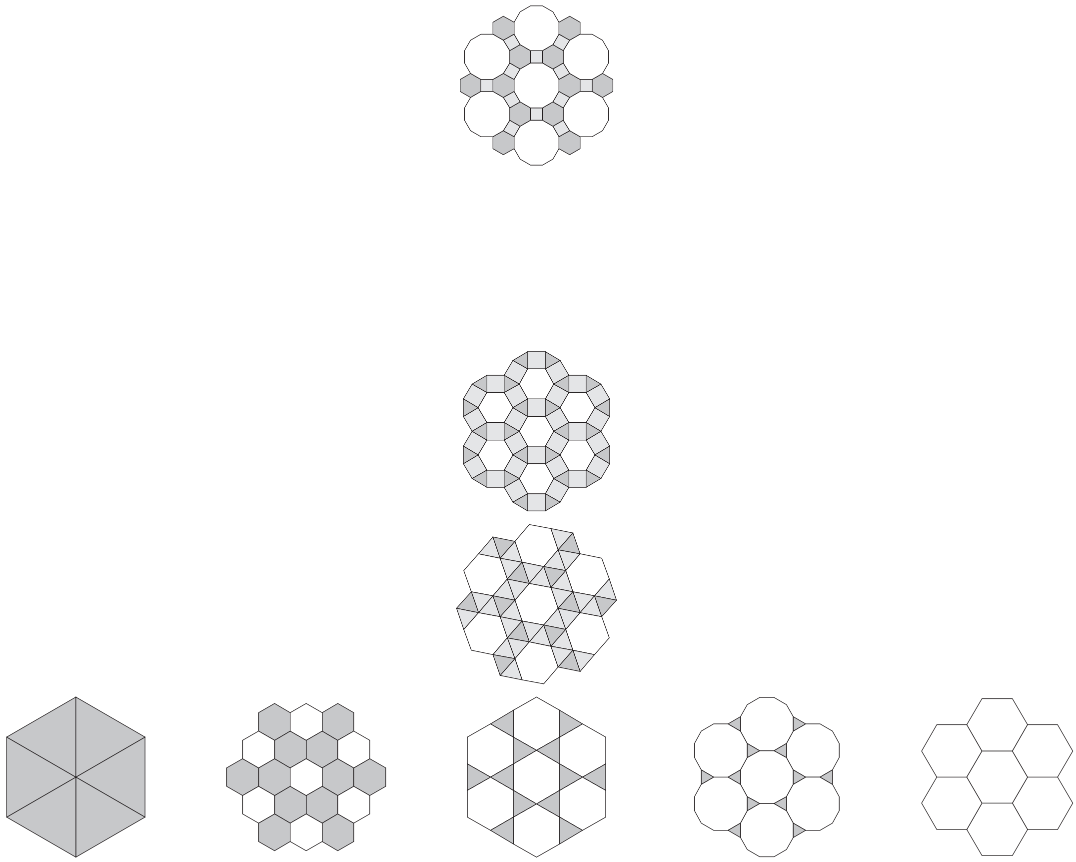


Figure 10 : Regular and semi-regular tilings of Class IV of TriHexagonal Array Symmetry.

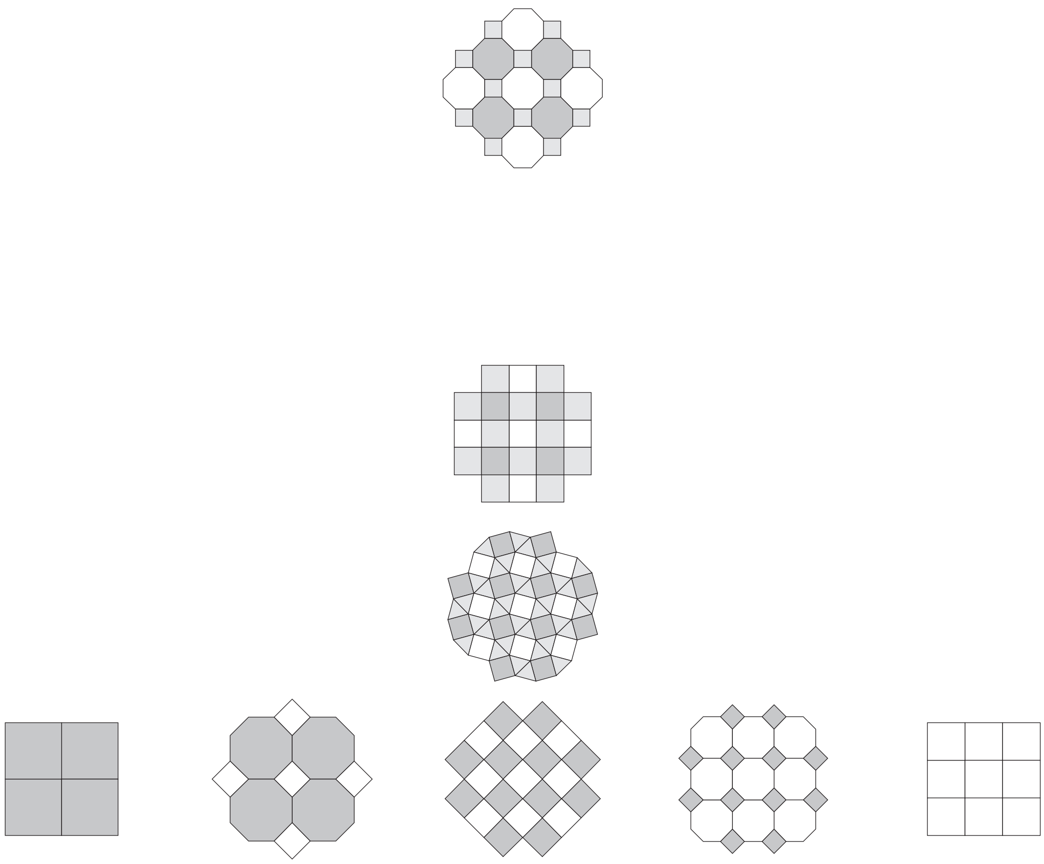


Figure 11 : Regular and semi-regular tilings of Class V of SquareSquare Array Symmetry.

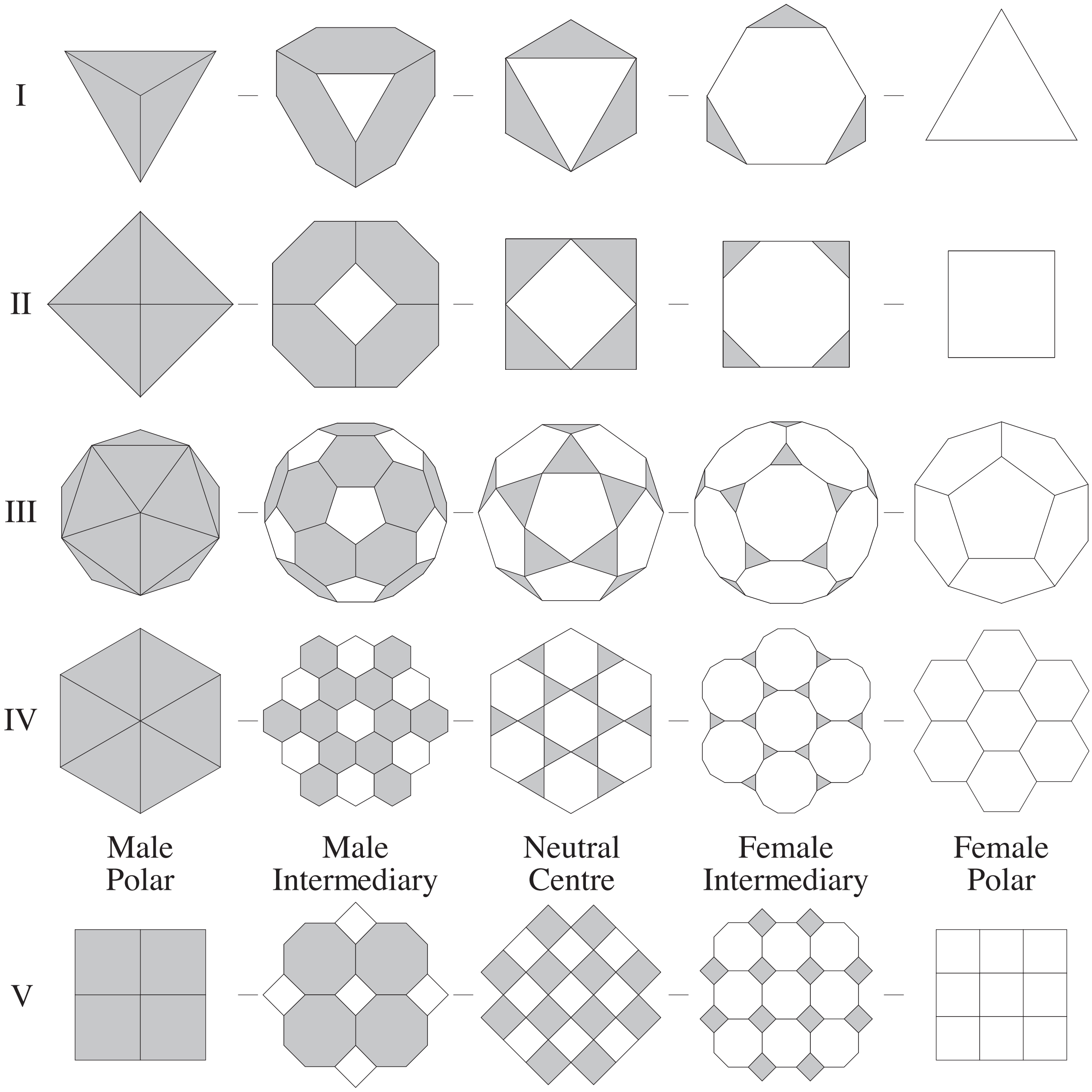


Figure 12 : Long Horizontal Sequences of the five classes showing the regular and semi-regular elements of First Degree. See also Fig 5.

THE META-PATTERN OVER ALL FIVE CLASSES

Reference to Fig 12 confirms that the positive polar element consists only of positive single-frequency faces, and the negative polar element consists only of negative single-frequency faces. The truncated positive intermediary element consists of double-frequency edge-jointed positive faces which cluster around and isolate the single-frequency negative faces, with three faces to a vertex. The truncated negative intermediary element consists of double-frequency edge-jointed negative faces which cluster around and isolate the single-frequency positive faces, with three faces to a vertex. The neutral central element of first degree alternates edge-jointed positive and negative single-frequency faces, so that a positive face is only edge-jointed to negative faces and vice versa, with positive faces vertex-jointed to one another as are negative faces; there are four faces to a vertex.

Fig 13 shows that part-way in the rotation of positive and negative faces of the central element of first degree towards forming the central element of second degree, the intermediary element of second degree occurs in two enantiomorphs. This snub or skew element alternates vertex-jointed positive and negative single-frequency faces, so that a positive face is only vertex-jointed to negative faces and vice versa. Paired triangular neutral faces separate the positive and negative faces, to which they are edge-jointed; there are five faces to a vertex.

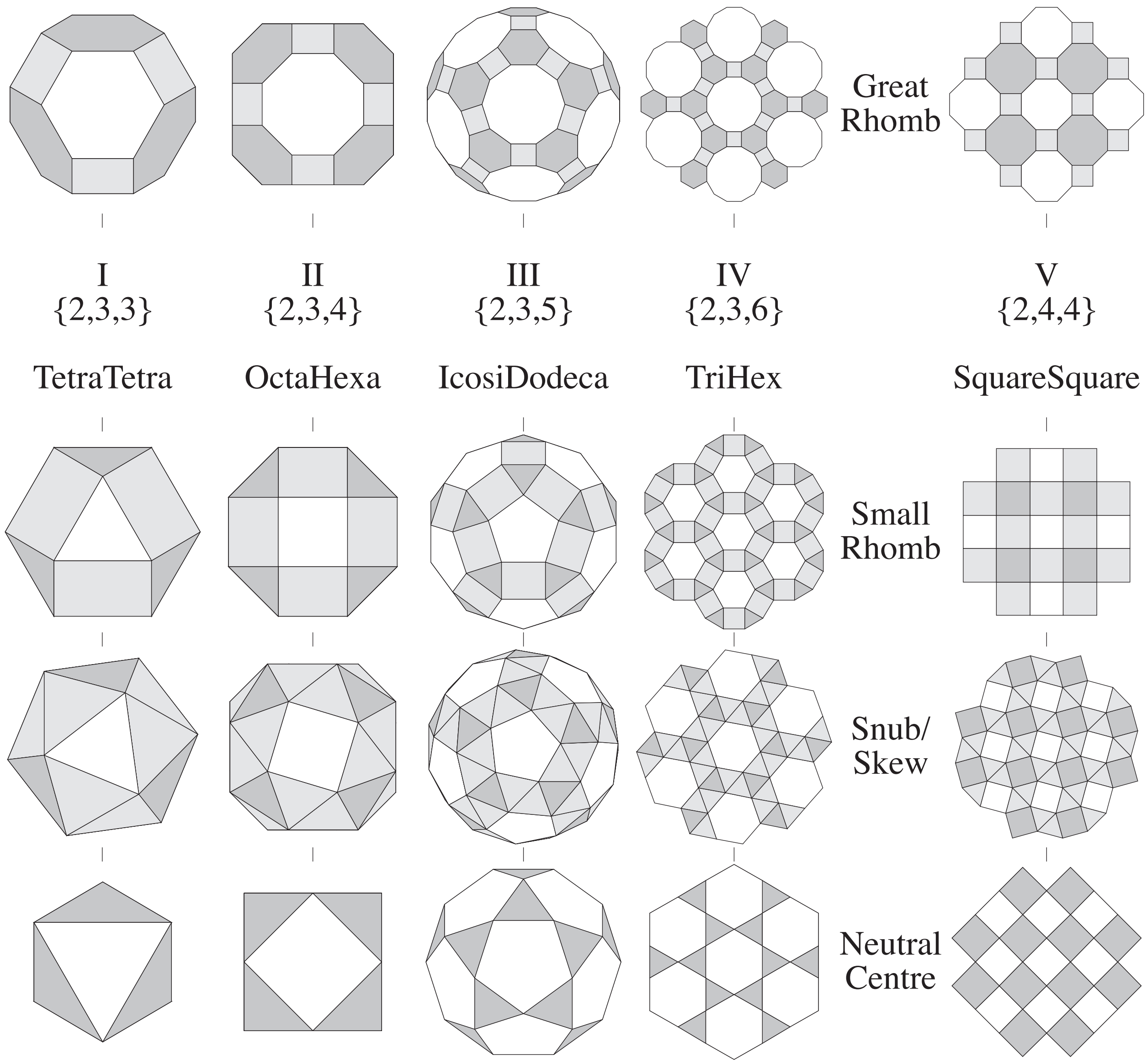


Figure 13 : Vertical Sequences of the five classes showing the regular and semi-regular Neutral Central elements of First, Second and Third Degree, together with the Intermediary elements of Second Degree. See also Fig 6.

The central element of second degree, or Small Rhomb of Fig 13, consists of single-frequency positive and negative faces, separated by square neutral faces to which they are edge-jointed, so that a positive face is only vertex-jointed to negative faces and vice versa. Positive and negative faces of the central element of first degree are rotated by half their central face angle, one set clockwise and the other counter. There are four faces to a vertex.

The central element of third degree, or Great Rhomb of Fig 13, is of double-frequency positive and negative faces and square neutral faces. Positive faces are edge-jointed to alternating negative and neutral faces; neutral to alternating positive and negative faces; and negative faces to alternating positive and neutral faces. There are three faces to a vertex.

These regularities are best appreciated by contemplation of the illustrations.

FIVE RHOMBIC DUAL ELEMENTS, AND OTHER POLYHEDRA

Three rhombic polyhedra, shown in Fig 14, are appended because of their importance as duals of the three key central polyhedra of first degree. All of their faces are neutral, and the cube reappears in Class I as the Dual Rhombic TetraTetrahedron or rhombic hexahedron. Rhombic duals of the key central patterns are also shown for the two-dimensional classes, with all faces being neutral. In Class IV this is a regular array of diamonds formed from paired equilateral triangles. In each class, there are two kinds of vertex, with three and three, three and four, three and five, three and six, and four and four faces respectively.

Class I (TetraTetrahedron)	Class II (OctaHexahedron)	Class III (IcosiDodecahedron)	Class IV (TriHexagon Array)
Dual Rhombic TetraTetrahedron *(Rhombic Hexahedron)*	Dual Rhombic OctaHexahedron *(Rhombic Dodecahedron)*	Dual Rhombic IcosiDodecahedron *(Rhombic Triacontahedron)*	Dual Rhombic TriHexagon Array *(Diamond Array)*
	Class V (SquareSquareArray)		
	Dual Rhombic SquareSquareArray *(Square Array)*		

Table 5: Rhombic Duals

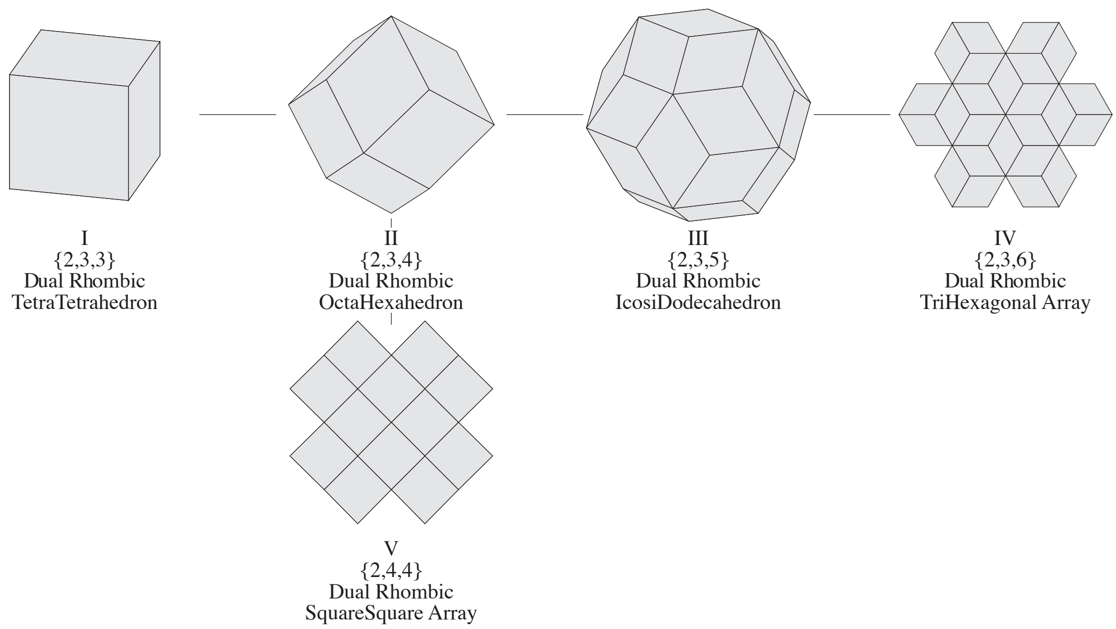


Figure 14 : Dual Rhombic elements of the quasi-regular neutral central elements of First Degree of Fig 1.

Other polyhedra can be integrated into the order, having regard to their symmetry patterns. For example, the Kepler-Poinsot polyhedra could be integrated as polar solids of Class III, where edges connect non-adjacent vertices. The joint solids of Stella Octangular (Tetrahedron-Tetrahedron), Octahedron-Cube, and Icosahedron-Dodecahedron represent a common central element of the three polyhedral classes, and equivalent tiling patterns (which are not semi-regular) are found for the other two classes by overlaying positive, negative and neutral symmetry arrays.

CONCLUSION

This paper presents a threefold classification of the regular and semi-regular polyhedra. It then extends that order to include a fourth and fifth class which generate the regular and all but one of the semi-regular tilings of the plane. Their development within the overall order advanced is consistent. Contemplation of Figs 7-11, which show each of the five classes, and Figs 12-13, which show the horizontal and vertical axes of the five classes, confirms this. Elements are classified according to their symmetry pattern and positioned relative to other elements within their class. In this process certain elements are repeated with different ascription of faces to symmetry axes. With certain regular exceptions, all faces are symmetrically disposed about their respective symmetry axes. The exceptions are the neutral faces of the snub and skew elements, where it is the paired triangles that are symmetrically disposed. The poles of each class are regular, and the centres are quasi-regular. Each class culminates in a great rhombic element. The regularity of the order allows properties of an element or aspect of

elements of one class to be generalised to the other classes (with common-sense being applied at the change in dimension), and this predictive ability is confirmed by experience and discovery. The order does not account for all thirty-two uniform colourings of the semi-regular tilings of the plane, nor is it intended to.

More generally, recognition is made of the orderly spatial structure underlying the specific expressions of regularity which constitute these beautiful polyhedra and surface patterns.

Classification of the regular and semi-regular polyhedra reveals meaningful aspects that illuminate the rich harmony of space that manifests these perfect forms.

***

1 The Kepler-Poinsot polyhedra are beyond the scope of the present enquiry, but could nevertheless be integrated into an extended classification according to their symmetries. The regular prisms and antiprisms are monoaxial solids, and thus not relevant to this classification which is of polyaxial solids. For somewhat similar reasons, I remain to be convinced the tiling of the plane of {2,2,2}-fold symmetry is a true polyaxial decentralised pattern, and exclude it and its derivatives and colourings from this treatment.

2 Critchlow, K., *Order in Space*, Thames and Hudson, London, 1969. See especially Appendix 1: A Periodic Arrangement of the Elements of Spatial Order.

3 Cundy, H.M. and Rollett, A.P., *Mathematical Models (Second Edition)*, Oxford University Press, London, 1961, p. 100.

4 Coxeter, H.S.M., *Regular Polytopes*, Dover Publications, New York, 1973, p.289.

5 Critchlow, op. cit. See pp. 34-37.

6 Coxeter, *Regular Polytopes*, op. cit., p.19.

7 Grünbaum, B. and Shephard, G.C., *Tilings and Patterns*, W.H. Freeman and Company, New York, 1987, p.110.

8 See Grünbaum and Shephard, ibid., Fig 2.1.5, p.63, for a discussion and illustrations of space-filling surface patterns. See also Critchlow, op. cit., pp. 60-61.

9 Meurant, R.C., *The Aesthetics of the Sacred ~ A Harmonic Geometry of Consciousness and Philosophy of Sacred Architecture*, Third Edition. The Opoutere Press, Auckland, 1989. This work also addresses two-dimensional centralised zonagonal clusters, which contrast with decentralised regular tilings of the plane.

10 Critchlow, op. cit., see in particular Appendix 2: *A Periodic Arrangement of the Multiple Single, Dual, and Triple All-Space Filling Solids Exhibiting a Periodicity of Eight, Punctuated by a 3,2,3 Symmetry.*

11 Meurant, R.C., *The Integral Space Habitation ~ Towards an Architecture of Space*. The Opoutere Press, Auckland, 1989.

12 The enantiomorphs of the Snubs and Skews are not considered as separate elements.

13 From an orthogonal orientation to the camera, these were first rotated by 15° clockwise about the vertical axis, and then 18° downwards about the left-right axis.

14 This necessitates a different orientation of the regulating cube for each of the three polyhedral classes.

15 See for example Lawlor, R., *Sacred Geometry - Philosophy and Practice*, Thames and Hudson, London, 1982.

16 Ghyka, M., *The Geometry of Art and Life*, Dover Publications, New York, 1977, Plates LXV, XIV, and LIX.

17 See Lund, F.M., *Ad Quadratum - A Study of the Geometrical Bases of Classic and Medieval Religious Architecture*, B.T. Batsford Ltd., London, 1921. Also see Brunés, T., *The Secrets of Ancient Geometry - and Its Use*, *vols. I & II*, Rhodos, Copenhagen, 1967; and Watts, D.J. and Watts, C.M., A Roman Apartment Complex, *Scientific American*, December 1986.

18 Excepting the neutral axes of the snub tetra tetrahedron $3.3.3.3^2$ (23133) of I:{2,3,3} (the icosahedron) and the skew squaresquare $4.3.4.3^2$ (23133) of IV: {2, 4, 4}, if the positive-negative axis distinction and thus colourings are ignored.

19 Male-Female polarity is a convenient means of structuring complexity, which accords with traditional symbolism. The Male and Female Tetrahedra are simply the two tetrahedra of dual orientation which are developed in a cube. Although alternative orientations of the icosahedron and dodecahedron in the cube exist, they are in contrast not dual to one another (i.e. icosahedron to icosahedron, etc.)

20 Holden, A., *Shapes, Space, and Symmetry*, Columbia University Press, New York, 1971, pp. 40-41.

21 Holden, ibid., p.48, illustrates the left- and right-handed tetrahedral symmetries of the faces of the icosahedron. Both enantiomorphs of the Snub Tetra tetrahedron are of course icosahedra.

22 Fuller, R.B., *Synergetics: Explorations in the Geometry of Thinking*, Vol.I, MacMillan Publishing Co., New York, 1975, Fig 460.08, p.193.

23 For example Edmondson merely repeats Fuller’s vector equilibrium contraction, and offers nothing new. See Edmondson, A.C., *A Fuller Explanation: the Synergetic Geometry of R. Buckminster Fuller*, Birkhaüser, Boston, 1987, Fig 11-1, p.160.

24 Cundy and Rollett, op. cit., p. 100.

25 Grünbaum and Shephard, op. cit., Fig 2.9.2, pp.104-6.

26 For example, a hybrid Class VI of {3,3,3}-fold symmetry may be advanced, which is generated orthogonal to the page from Class I in the metapattern shown above. (Intriguingly, the three extensions of the polyhedral pattern to generate two-dimensional Classes IV, V, and VI then occur along each of the three axes respectively). Using Grünbaum and Shephard’s symbols, this TriTri class contains 3.3.3.3.3.3 (212121), 3.6.3.6 (2313), and 6.6.6 (231) as central tilings of first, second and third degree respectively. With a 3-fold neutral axis, the neutral faces are fully expanded into hexagons. The skew tiling is $3.3.3.3^3$ (231333), containing two-frequency triangular neutral faces, which occur as contracted hexagons. The vertical sequence is complete. But the remaining elements of the horizontal sequence do not exist. More critically, the positive and negative symmetry systems are not dual, and the central element should not therefore be considered quasi-regular. Both poles are virtual triangular arrays of triangles, which are not possible without leaving triangular holes; or 3^6 (111111) and (222222) could be taken as male and female poles, but then the requirements for conservation of spatial array and orientation would need to be relaxed. Similarly, it is tempting to postulate 3.3^5 (211111) and (122222) as both male and female intermediaries; but then the respective polar faces of that intermediary are not of double frequency, and their spatial array is not conserved. Thus although elements could be found by relaxing the constraints on the definition, I prefer not to.